J. Fluid Mech. (1968), *vol.* 32, *part* 2, *pp.* 393–398
Printed in Great Britain

The Rayleigh–Jeffreys problem with boundary slab of finite conductivity

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(Received 11 September 1967)

Linear perturbation analysis is applied to the problem of the onset of convection in a horizontal layer of fluid heated uniformly from below, when the fluid is bounded below by a rigid plate of infinite conductivity and above by a solid layer of finite conductivity and finite thickness. The critical Rayleigh number and wave-number are found for various thickness ratios and thermal conductivity ratios. Both numbers are reduced by the presence of a boundary of finite (rather than infinite) conductivity in qualitative agreement with the observation of Koschmieder (1966).

1. Introduction

The problem of determining the conditions for the onset of convection induced by buoyancy effects resulting from the heating from below of a horizontal layer of a viscous fluid, received its first theoretical treatment from Rayleigh (1916) and Jeffreys (1926). (The problem has been referred to as the Bénard problem, but this term is more appropriate for the related surface-tension problem.) Many subsequent authors have extended the theory, a standard reference for which is Chandrasekhar (1961).

Until recently attention has been concentrated on the case where both upper and lower boundaries are of infinite thermal conductivity. Sparrow, Goldstein & Jonsson (1964) extended the classical Rayleigh–Jeffreys analysis to allow for a 'radiation' type condition at a boundary, while Hurle, Jakeman & Pike (1967) considered a layer of fluid bounded below and above by the same solid material, of finite thermal conductivity, extending to infinity in each vertical direction. In the present paper we treat a configuration more practical than that of Hurle *et al.* Our model is as follows. The upper surface of the fluid is adjacent to the lower side of a horizontal slab of solid material whose upper side is in turn bounded by a medium of infinite conductivity. The lower boundary of the fluid is assumed to be an infinitely conducting rigid plate. (The analysis below could, if desired, easily be adapted for the case where this plate was replaced by a second slab of finite conductivity.)

Most experiments on the Rayleigh–Jeffreys problem have been designed to ensure good heat conduction on the boundaries. An exception is the experiment described by Koschmieder (1966) who, in order to facilitate visual observation of the convection patterns, used as a lid over the fluid a plate of glass whose thermal conductivity was only an order of magnitude greater than that of the fluid. Koschmieder found that when the Rayleigh number was continuously increased beyond the critical value for the onset of convection, the wave-number of the convection pattern decreased. This decrease, which is also evident from photographs published by Schmidt & Silveston (1959), is in disagreement with the theory (for infinitely conducting boundaries) of Platzman (1965) or Schlüter, Lortz & Busse (1965). As was pointed out by Dr Koschmieder in a private communication to the author, the decrease in wave-number is apparently the result of increasingly less effective transfer of heat perturbations in the media (glass and cooling fluid) above the observed fluid layer in comparison with that in this convecting fluid. The present work was undertaken in an attempt to throw some light, as far as a linear theory can, on this phenomenon.

2. Analysis

We consider a fluid layer of depth d overlain by a solid layer of thickness d'. Thus, with z indicating distances vertically upwards, the fluid will occupy the region $0 < z < d$ and the solid the region $d < z < d+d'$. The planes $z = 0$ and $z = d+d'$ are held at uniform temperatures T_0 and T_1 respectively, and each plane is assumed to be a perfect thermal conductor. In the steady state the fluid velocity is then zero and the temperature distribution is given by

$$T_* = \begin{cases} T_0-\beta z & \text{in} \quad 0 \leqslant z \leqslant d, \\ T_1+\beta'(d+d'-z) & \text{in} \quad d \leqslant z \leqslant d+d', \end{cases}$$

where

$$\beta = \frac{K'(T_0-T_1)}{K'd+Kd'}, \quad \beta' = \frac{K(T_0-T_1)}{K'd+Kd'}$$

are the adverse temperature gradients in the fluid (of thermal conductivity K) and the solid (of thermal conductivity K') respectively.

The analysis now follows that of Chandrasekhar (1961). The fluid is assumed to be of the Boussinesq type (quasi-incompressible, and otherwise with constant fluid properties). Perturbations from the steady-state solution are considered, and the governing differential equations are linearized. An expansion in normal modes, involving a separation of variables, is performed. The following equations for a steady neutral disturbance are obtained.

In $0 \leqslant z \leqslant d$,

$$\nu(d^2/dz^2-k^2)^2 W - g\alpha k^2 \Theta = 0, \tag{1}$$

and

$$\kappa(d^2/dz^2-k^2)\Theta + \beta W = 0, \tag{2}$$

while in $d \leqslant z \leqslant d+d'$,

$$(d^2/dz^2-k^2)\Theta' = 0. \tag{3}$$

Here α, ν and κ are the thermal expansion coefficient, kinematic viscosity and thermal diffusivity of the fluid, g is the gravitational acceleration, and k is the horizontal wave-number of the disturbance. $W(z)$ and $\Theta(z)$ give the variation with z of the vertical component of the fluid velocity, and the temperature perturbation in the fluid, respectively. In the solid the corresponding quantities are denoted by primes.

The thermal boundary conditions are

$$\Theta = 0 \quad \text{at} \quad z = 0, \tag{4}$$

$$\Theta = \Theta' \quad \text{and} \quad K\,\partial\Theta/\partial z = K'\,\partial\Theta'/\partial z \quad \text{at} \quad z = d, \tag{5}$$

$$\Theta' = 0 \quad \text{at} \quad z = d+d'. \tag{6}$$

(It seems that in the paper of Hurle *et al.* (1967), at this stage of the analysis and in their table of results, it is the thermal conductivity K rather than the thermal diffusivity κ which should be used.)

The solution of (3) subject to the condition (6) is

$$\Theta' = C \sinh k(d+d'-z),$$

where C is a constant. Conditions (5) then imply that

$$\frac{1}{\Theta}\frac{\partial\Theta}{\partial z} = \frac{K}{K'}\,k \coth kd \quad \text{at} \quad z = d. \tag{7}$$

The no-slip condition and the equation of continuity lead to the conditions

$$W = \partial W/\partial z = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad z = d. \tag{8}$$

Our problem is now reduced to the solution of (1) and (2) subject to the boundary conditions (4), (7) and (8).

If we let $W_1 = Wd/\pi\nu$, $\Theta_1 = \Theta\pi\kappa/\beta d\nu$, $b = kd/\pi$, $\xi = \pi z/d$, and write D for $d/d\xi$, the equations take the form:

$$(D^2-b^2)^2 W_1 = Rb^2\Theta_1, \tag{9}$$

$$(D^2-b^2)\Theta_1 = -W_1, \tag{10}$$

with

$$W_1 = 0, \quad DW_1 = 0, \quad \Theta_1 = 0 \quad \text{at} \quad z = 0, \tag{11}$$

$$W_1 = 0, \quad DW_1 = 0, \quad D\Theta_1 = -(L/\pi)\Theta_1 \quad \text{at} \quad z = \pi, \tag{12}$$

where the Rayleigh number

$$R^* = \pi^4 R = g\alpha\beta d^4/\kappa\nu, \tag{13}$$

and the heat transfer parameter

$$L = \pi b(K'/K)\coth(\pi b d'/d). \tag{14}$$

This system of equations is a subcase of that solved by Nield (1967), but now L is no longer a free parameter. The resulting eigenvalue equation may be written:

$$\begin{vmatrix} \Sigma E_n F_n^2 & \Sigma(-1)^n E_n F_n^2 & Rb^2\Sigma(-1)^n E_n F_n \\ \Sigma(-1)^n E_n F_n^2 & \Sigma E_n F_n^2 & Rb^2\Sigma E_n F_n \\ \Sigma(-1)^n E_n F_n & \Sigma E_n F_n & Rb^2\Sigma E_n - \tfrac{1}{2}(L+\pi b\coth \pi b) \end{vmatrix} = 0, \tag{15}$$

where $F_n = n^2+b^2$, $E_n = n^2/(F_n^4 - Rb^2F_n)$, the sums are all from $n = 1$ to $n = \infty$, and L is given by (14).

For given values of the parameters K'/K and d'/d, equation (15) determines the reduced Rayleigh number R as a function of the disturbance wave-number b. (The function is multivalued, there being one branch for each mode of instability.)

The minimum value of R as b varies is then the critical value R_c, which, for the lowest mode, corresponds to the onset of convection. The numerical calculation is straightforward.

3. Results and discussion

In tables 1 (*a*) and 1 (*b*) are presented values of the critical Rayleigh number $R_c^* = \pi^4 R_c$ and the corresponding critical wave-number $a_c = \pi b_c$ for the most unstable mode. As expected, both R_c and a_c increase with K'/K and decrease with increase of d'/d. The same trend is seen in the values given in tables 2 (*a*) and 2 (*b*) for the second mode of instability.

	d'/d						
K'/K	0	0·01	0·03	0·1	0·3	1	⩾10
0	—	1295·8	1295·8	1295·8	1295·8	1295·8	1295·8
0·01	1707·8	1398·5	1337·9	1309·8	1301·4	1299·5	1299·4
0·03	1707·8	1497·6	1398·7	1334·9	1312·1	1306·6	1306·5
0·1	1707·8	1607·1	1508·3	1400·4	1345·1	1329·9	1329·6
0·3	1707·8	1667·1	1607·3	1500·3	1414·0	1384·1	1383·4
1	1707·8	1694·6	1670·9	1610·3	1529·7	1493·4	1492·7
3	1707·8	1703·3	1694·6	1668·2	1623·4	1599·3	1598·9
10	1707·8	1706·5	1704·0	1694·9	1678·1	1668·1	1668·0
100	1707·8	1707·6	1707·4	1706·4	1704·6	1703·4	1703·4
∞	1707·8	1707·8	1707·8	1707·8	1707·8	1707·8	1707·8

TABLE 1 (*a*). Values of the critical Rayleigh number $R_c^* = \pi^4 R$ for the lowest mode of instability

	d'/d						
K'/K	0	0·01	0·03	0·1	0·3	1	⩾10
0	—	2·553	2·553	2·553	2·553	2·553	2·553
0·01	3·117	2·753	2·639	2·582	2·564	2·557	2·556
0·03	3·117	2·901	2·750	2·632	2·582	2·566	2·565
0·1	3·117	3·028	2·914	2·751	2·641	2·596	2·594
0·3	3·117	3·084	3·028	2·900	2·753	2·669	2·665
1	3·117	3·107	3·087	3·029	2·910	2·819	2·815
3	3·117	3·113	3·107	3·084	3·027	2·967	2·964
10	3·117	3·115	3·111	3·106	3·086	3·063	3·062
100	3·117	3·116	3·116	3·115	3·114	3·110	3·110
∞	3·117	3·117	3·117	3·117	3·117	3·117	3·117

TABLE 1 (*b*). Values of the corresponding critical wave-number $a_c = \pi b_c$ for the lowest mode of instability

	d'/d (*a*)				d'/d (*b*)			
K'/K	0	0·1	1	⩾10	0	0·1	1	⩾10
0	—	15,278	15,278	15,278	—	4·91	4·91	4·91
0·1	17,610	15,654	15,467	15,467	5·37	5·01	4·94	4·94
1	17,610	16,803	16,382	16,382	5·37	5·24	5·11	5·11
10	17,610	17,492	17,421	17,380	5·37	5·35	5·32	5·31
∞	17,610	17,610	17,610	17,610	5·37	5·37	5·37	5·37

TABLES 2 (*a*) and 2 (*b*). Values of the critical Rayleigh number R_c^* and the corresponding critical wave-number a_c, for the second mode of instability

The published experimental results which are most relevant to the present theory are those of Koschmieder (1966), but his experiment was designed primarily for the visual observation of the convection patterns and not for obtaining precise quantitative data. By the time the motion of his fluid was sufficiently well developed to be visible, it is probable that the Rayleigh number was substantially above the critical value. Thus the agreement between the values $R = 1730$ and $a = 3{\cdot}2$ from Koschmieder's experiment for $K'/K = 8$ and $d'/d = 0{\cdot}3$, and the value $R_c = 1650$ and $a_c = 3{\cdot}08$ from the present theory, is as good as could be expected.

When Koschmieder continued to increase the rate of heating of the fluid, he observed that the wave-number a of the convective motion declined from the value 3·2 to 2·3, but when the latter value of a had been reached the motion had become markedly unsymmetrical. The reduction in wave-number cannot be explained merely by noting that higher order modes of instability become increasingly effective as the Rayleigh number is increased, since the wave-number is larger for the higher order modes (for boundaries of given constant conductivity). It appears that the observed effect is a result of the effective decrease in the appropriate value of our ratio K'/K (or effective increase in d'/d) as the water circulating above Koschmieder's glass plate became less and less effective in transporting away the increased heat flux. In other words, our assumption that the surface $z = d+d'$ is perfectly conducting becomes increasingly less appropriate. Presumably the same is true, but to a lesser extent, at the boundary $z = 0$, which in Koschmieder's experiment was the upper surface of a copper plate. A proper explanation of the reduction in wave-number with increase in heat flux must, of course, invoke non-linear considerations, but our linear theory does at least make plausible a reduction in wave-number from 3·1 to 2·5 (or to a smaller value if the conductivity of the material below the fluid layer is given a finite value rather than an infinite one). Further experimental and theoretical work is highly desirable.

The author is grateful to Dr Koschmieder for drawing his attention to this problem and the likely explanation.

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